

(p, q) -Rogers-Szegö polynomial and the (p, q) -oscillator^{1 2}

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Dedicated to the memory of Professor Alladi Ramakrishnan

Abstract

A (p, q) -analogue of the classical Rogers-Szegö polynomial is defined by replacing the q -binomial coefficient in it by the (p, q) -binomial coefficient corresponding to the definition of (p, q) -number as $[n]_{p,q} = (p^n - q^n)/(p - q)$. Exactly like the Rogers-Szegö polynomial is associated with the q -oscillator algebra it is found that the (p, q) -Rogers-Szegö polynomial is associated with the (p, q) -oscillator algebra.

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1 Introduction

The q -oscillator algebra plays a central role in the physical applications of quantum groups (for a review of quantum groups and their applications see, *e.g.*, [1]-[3]). It was used [4]-[7] to extend the Jordan-Schwinger realization of the $sl(2)$ algebra in terms of harmonic oscillators to the q -analogue of the universal enveloping algebra of $sl(2)$, namely, $U_q(sl(2))$. In order to extend this q -oscillator realization of $U_q(sl(2))$ to the two-parameter quantum algebra $U_{p,q}(gl(2))$ the (p, q) -oscillator algebra was introduced in [8] (see also [9, 10]).

Heine's q -number, or the basic number,

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (1.1)$$

is well known in the mathematics literature. The (p, q) -oscillator necessitated the introduction of the (p, q) -number, or the twin-basic number,

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad (1.2)$$

a natural generalization of the q -number, such that

$$\lim_{p \rightarrow 1} [n]_{p,q} \longrightarrow [n]_q. \quad (1.3)$$

With the introduction of this (p, q) -number, the essential elements of the (p, q) -calculus, namely, (p, q) -differentiation, (p, q) -integration, and the (p, q) -exponential, were also studied in [8]. This led to a more detailed study of (p, q) -hypergeometric series and (p, q) -special functions [11]-[13]. Meanwhile, the (p, q) -binomial coefficients, (p, q) -Stirling numbers, and the (p, q) -binomial theorem for noncommutative operators had been studied [14, 15] in the analysis of certain physical problems. Interestingly, in the same year 1991 the (p, q) -number was introduced in the mathematics literature in connection with set partition statistics [16]. A very general formalism of deformed hypergeometric functions has been developed in [17]. Some applications of (p, q) -hypergeometric series in the context of two-parameter quantum groups can be found in [18, 19].

It is noted in [4] that the classical Rogers-Szegő polynomials provide a basis for a coordinate representation of the q -oscillator. Several aspects of this close connection between the q -oscillator algebra and the Rogers-Szegő

polynomials, and the related continuous q -Hermite polynomials, have been analysed in detail later (see, *e.g.*, [20]-[24]). In this paper, after a brief review of the known connection between the Rogers-Szegö polynomial and the q -oscillator, we shall define a (p, q) -Rogers-Szegö polynomial and show that it is connected with the (p, q) -oscillator.

As explained below in section 4, it is not possible to rewrite a (p, q) -hypergeometric series, or a (p, q) -analogue of a q -function, as a regular q -hypergeometric series or a q -function routinely by rescaling the independent variable. Particularly, this is not possible in the case of the (p, q) -Rogers-Szegö polynomial considered here.

2 Harmonic oscillator

The harmonic oscillator is associated with the creation (or raising) operator \hat{a}_+ , the annihilation (or lowering) operator \hat{a}_- , and the number operator \hat{n} satisfying the algebra

$$[\hat{n}, \hat{a}_+] = \hat{a}_+, \quad [\hat{n}, \hat{a}_-] = -\hat{a}_-, \quad [\hat{a}_-, \hat{a}_+] = 1, \quad (2.1)$$

where $[\hat{A}, \hat{B}]$ stands for the commutator $\hat{A}\hat{B} - \hat{B}\hat{A}$. Note that

$$\hat{n} = \hat{a}_+ \hat{a}_-. \quad (2.2)$$

Let

$$h_n(x) = (1+x)^n, \quad \psi_n(x) = \frac{1}{\sqrt{n!}} h_n(x). \quad (2.3)$$

It follows that

$$\frac{d}{dx} \psi_n(x) = \sqrt{n} \psi_{n-1}(x), \quad (2.4)$$

$$(1+x) \psi_n(x) = \sqrt{n+1} \psi_{n+1}(x), \quad (2.5)$$

$$(1+x) \frac{d}{dx} \psi_n(x) = n \psi_n(x), \quad (2.6)$$

$$\frac{d}{dx} ((1+x) \psi_n(x)) = (n+1) \psi_n(x). \quad (2.7)$$

Thus, it is clear that the set $\{\psi_n(x) | n = 0, 1, 2, \dots\}$ forms a basis for the following Bargmann-Fock realization of the oscillator algebra (2.1):

$$\hat{a}_+ = (1+x), \quad \hat{a}_- = \frac{d}{dx}, \quad \hat{n} = (1+x) \frac{d}{dx}. \quad (2.8)$$

It is to be noted that equations (2.5) and (2.6) are, respectively, the recurrence relation and the differential equation for $\psi_n(x)$.

Let $\{\hat{a}_-^{(1)}, (\hat{a})_+^{(1)}, \hat{n}^{(1)}\}$ and $\{\hat{a}_-^{(2)}, \hat{a}_+^{(2)}, \hat{n}^{(2)}\}$ be two sets of oscillator operators each satisfying the algebra (2.1) and commuting with each other. Then, for the generators of $sl(2)$ satisfying the Lie algebra,

$$[x_0, x_+] = x_+, \quad [x_0, x_-] = -x_-, \quad [x_-, x_+] = 2x_0, \quad (2.9)$$

one has the Jordan-Schwinger realization

$$x_+ = \hat{a}_+^{(1)} \hat{a}_-^{(2)}, \quad x_- = \hat{a}_+^{(2)} \hat{a}_-^{(1)}, \quad x_0 = \frac{1}{2} (\hat{n}^{(1)} - \hat{n}^{(2)}). \quad (2.10)$$

3 q -Oscillator and the Rogers-Szegö polynomial

When $U(sl(2))$, the universal enveloping algebra of $sl(2)$, is q -deformed the resulting $U_q(sl(2))$ has generators $\{X_-, X_+, X_0\}$ satisfying the algebra

$$\begin{aligned} [X_0, X_+] &= X_+, & [X_0, X_-] &= -X_-, \\ X_- + X_- - q^{-1} X_- X_+ &= \frac{1 - q^{2X_0}}{1 - q} = [2X_0]_q. \end{aligned} \quad (3.1)$$

The q -oscillator is associated with the annihilation operator \hat{A}_- , creation operator \hat{A}_+ and the number operator \hat{N} satisfying the algebra

$$[\hat{N}, \hat{A}_-] = -\hat{A}_-, \quad [\hat{N}, \hat{A}_+] = \hat{A}_+, \quad \hat{A}_- \hat{A}_+ - q \hat{A}_+ \hat{A}_- = 1. \quad (3.2)$$

It should be noted that in this case $\hat{N} \neq \hat{A}_+ \hat{A}_-$. Instead we have

$$\hat{A}_+ \hat{A}_- = \frac{1 - q^{\hat{N}}}{1 - q} = [\hat{N}]_q, \quad (3.3)$$

and

$$\hat{A}_- \hat{A}_+ = \frac{1 - q^{\hat{N}+1}}{1 - q} = [\hat{N} + 1]_q, \quad (3.4)$$

Now, let $\{\hat{A}_-^{(1)}, (\hat{A})_+^{(1)}, \hat{N}^{(1)}\}$ and $\{\hat{A}_-^{(2)}, \hat{A}_+^{(2)}, \hat{N}^{(2)}\}$ be two sets of q -oscillator operators each satisfying the algebra (3.2) and commuting with each other.

Then, taking

$$X_+ = \hat{A}_+^{(1)} q^{-\hat{N}^{(2)}/2} \hat{A}_-^{(2)}, \quad X_- = \hat{A}_+^{(2)} q^{-\hat{N}^{(2)}/2} \hat{A}_-^{(1)}, \quad X_0 = \frac{1}{2} (\hat{N}^{(1)} - \hat{N}^{(2)}), \quad (3.5)$$

we get a Jordan-Schwinger-type realization of the $U_q(sl(2))$ (3.1).

Now, we have to recall some definitions from the theory of q -series. The q -shifted factorial is defined as

$$(a; q)_n = \begin{cases} 1, & \text{for } n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & \text{for } n = 1, 2, \dots \end{cases} \quad (3.6)$$

The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \quad (3.7)$$

For more details on q -series see [25]. With the definition

$$[0]_q! = 1, \quad [n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q, \quad \text{for } n = 1, 2, \dots, \quad (3.8)$$

we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad (3.9)$$

and

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad (3.10)$$

The Rogers-Szegö polynomial is defined as

$$H_n(x; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \quad (3.11)$$

This can be naturally expected to be related to the basis of a realization of the q -oscillator since

$$\lim_{q \rightarrow 1} H_n(x; q) = h_n(x), \quad (3.12)$$

and the q -oscillator becomes the ordinary oscillator in the limit $q \rightarrow 1$.

To exhibit the relation between $H_n(x; q)$ and the q -oscillator we shall closely follow [22], though our treatment is slightly different. Let us define

$$\psi_n(x; q) = \frac{1}{\sqrt{[n]_q!}} H_n(x; q) = \frac{1}{\sqrt{[n]_q!}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \quad (3.13)$$

The Jackson q -difference operator is defined by

$$\hat{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \quad (3.14)$$

It is straightforward to see that

$$\hat{D}_q \psi_n(x; q) = \sqrt{[n]_q} \psi_{n-1}(x; q). \quad (3.15)$$

The q -binomial coefficients obey the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q - (1-q^n) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \quad (3.16)$$

From this it follows that $\psi_n(x; q)$ satisfies the recurrence relation

$$\sqrt{[n+1]_q} \psi_{n+1}(x; q) = (1+x) \psi_n(x; q) - x(1-q) \sqrt{[n]_q} \psi_{n-1}(x; q). \quad (3.17)$$

Using (3.15) we can write this relation as

$$[(1+x) - (1-q)x\hat{D}_q] \psi_n(x; q) = \sqrt{[n+1]_q} \psi_{n+1}(x; q). \quad (3.18)$$

Thus, it is seen from (3.15) and (3.18) that the set of polynomials $\{\psi_n(x; q) | n = 0, 1, 2, \dots\}$ provides a basis for a realization of the q -oscillator algebra (3.2) as follows. Let us define the number operator \hat{N} formally as

$$\hat{N} \psi_n(x; q) = n \psi_n(x; q). \quad (3.19)$$

Note that

$$\hat{D}_q^{n+1} \psi_n(x; q) = 0, \quad (3.20)$$

and $\hat{D}_q^m \psi_n(x; q) \neq 0$ for any $m < n+1$. Then,

$$\hat{A}_- \psi_n(x; q) = \hat{D}_q \psi_n(x; q) = \sqrt{[n]_q} \psi_{n-1}(x; q), \quad (3.21)$$

$$\begin{aligned} \hat{A}_+ \psi_n(x; q) &= [(1+x) - (1-q)x\hat{D}_q] \psi_n(x; q) \\ &= \sqrt{[n+1]_q} \psi_{n+1}(x; q), \end{aligned} \quad (3.22)$$

$$\hat{A}_+ \hat{A}_- \psi_n(x; q) = [n]_q \psi_n(x; q) = [\hat{N}]_q \psi_n(x; q), \quad (3.23)$$

$$\hat{A}_- \hat{A}_+ \psi_n(x; q) = [n+1]_q \psi_n(x; q) = [\hat{N} + 1]_q \psi_n(x; q). \quad (3.24)$$

From this one can easily verify that the relations in (3.2) are satisfied by $\{\hat{A}_+, \hat{A}_-, \hat{N}\}$. It may be noted that these relations (3.21)-(3.24) are the q -generalizations of the harmonic oscillator relations (2.4)-(2.7) to which they reduce in the limit $q \rightarrow 1$. Substituting the explicit expressions for \hat{A}_+ and \hat{A}_- in (3.23) we get the q -differential equation for $\psi_n(x; q)$ (or $H_n(x; q)$; see [22]):

$$\left((1-q)x\hat{D}_q^2 - (1+x)\hat{D}_q + [n]_q\right) \psi_n(x; q) = 0. \quad (3.25)$$

which reduces to (2.6) in the limit $q \rightarrow 1$.

4 (p, q)-Oscillator and the (p, q)-Rogers-Szegő polynomial

A genuine two-parameter quantum deformation exists only for $U(gl(2))$ and not for $U(sl(2))$. The two-parameter deformation of $U(gl(2))$ leads to $U_{p,q}(gl(2))$ which is generated by $\{\hat{\mathcal{X}}_0, \hat{\mathcal{X}}_+, \hat{\mathcal{X}}_-\}$ satisfying the commutation relations

$$\begin{aligned} [\hat{\mathcal{X}}_0, \hat{\mathcal{X}}_+] &= \hat{\mathcal{X}}_+, & [\hat{\mathcal{X}}_0, \hat{\mathcal{X}}_-] &= -\hat{\mathcal{X}}_-, \\ \hat{\mathcal{X}}_+ \hat{\mathcal{X}}_- - (pq)^{-1} \hat{\mathcal{X}}_- \hat{\mathcal{X}}_+ &= \frac{p^{2\hat{\mathcal{X}}_0} - q^{2\hat{\mathcal{X}}_0}}{p - q} = [2\hat{\mathcal{X}}_0]_{p,q}, \end{aligned} \quad (4.1)$$

and a central element $\hat{\mathcal{Z}}$ which we shall ignore for the present purpose. Here, $[\]_{p,q}$ is as defined in (1.2).

To get an oscillator realization of the algebra (4.1) we need the (p, q) -oscillator algebra defined by

$$[\hat{\mathcal{N}}, \hat{\mathcal{A}}_+] = \hat{\mathcal{A}}_+, \quad [\hat{\mathcal{N}}, \hat{\mathcal{A}}_-] = -\hat{\mathcal{A}}_-, \quad \hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ - q \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- = p^{\hat{\mathcal{N}}}, \quad (4.2)$$

where $\{\hat{\mathcal{A}}_+, \hat{\mathcal{A}}_-, \hat{\mathcal{N}}\}$ are, respectively, the creation, annihilation, and number operators. In this case

$$\hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- = \frac{p^{\hat{\mathcal{N}}} - q^{\hat{\mathcal{N}}}}{p - q} = [\hat{\mathcal{N}}]_{p,q}, \quad \hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ = \frac{p^{\hat{\mathcal{N}}+1} - q^{\hat{\mathcal{N}}+1}}{p - q} = [\hat{\mathcal{N}} + 1]_{p,q}. \quad (4.3)$$

Note the symmetry of this relation under the exchange of p and q . So, the last relation in (4.2) can also be taken, equivalently, as

$$\hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ - p \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- = q^{\hat{\mathcal{N}}}. \quad (4.4)$$

The (p, q) -oscillator unifies several special cases of q -oscillators, including the q -fermion oscillator [26]. Now, we shall show that a (p, q) -deformation of the Rogers-Szegő polynomial can be used for a realization of the (p, q) -oscillator algebra exactly in the same way as the classical Rogers-Szegő polynomial is used for a realization of the q -oscillator algebra as seen above. To this end we proceed as follows.

First, let us recall some essential elements of the (p, q) -series (for more details see [12, 13]). The (p, q) -shifted factorial is defined by

$$(a, b; p, q)_n = \begin{cases} 1, & \text{for } n = 0, \\ \prod_{k=0}^{n-1} (ap^k - bq^k), & \text{for } n = 1, 2, \dots \end{cases} \quad (4.5)$$

Note that

$$(a, b; p, q)_n = a^n p^{n(n-1)/2} (b/a; q/p)_n. \quad (4.6)$$

In view of this, it is not possible to rewrite a (p, q) -hypergeometric series, or a (p, q) -analogue of a q -function, routinely as a q/p -hypergeometric series or a q/p -function, with the same or a rescaled independent variable, unless the factors depending on a and p in the numerator and the denominator cancel in each term, or are such that the uncanceled factor in each term is of the same power as the independent variable. The (p, q) -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{(p, q; p, q)_n}{(p, q; p, q)_k (p, q; p, q)_{n-k}}. \quad (4.7)$$

With the definition

$$[0]_{p,q}! = 1, \quad [n]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \dots [2]_{p,q} [1]_{p,q}, \quad \text{for } n = 1, 2, \dots, \quad (4.8)$$

we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!} \quad (4.9)$$

and

$$\lim_{p \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_q. \quad (4.10)$$

Let us now define the (p, q) -Rogers-Szegő polynomial as

$$H_n(x; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k, \quad (4.11)$$

and take

$$\psi_n(x; p, q) = \frac{1}{\sqrt{[n]_{p,q}!}} H_n(x; p, q) = \frac{1}{\sqrt{[n]_{p,q}!}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k. \quad (4.12)$$

Note that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p}, \quad (4.13)$$

and the presence of the factor p^{-k^2} makes it impossible to rescale x in any way and hence rewrite the (p, q) -Rogers-Szegö polynomial $H_n(x; p, q)$ as a regular Rogers-Szegö polynomial (3.11). Recalling the definition of the (p, q) -difference operator [8],

$$\hat{D}_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad (4.14)$$

it is seen that

$$\hat{D}_{p,q} \psi_n(x; p, q) = \sqrt{[n]_{p,q}} \psi_{n-1}(x; p, q). \quad (4.15)$$

The (p, q) -analogue of (3.16) is given by

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} - (p^n - q^n) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}. \quad (4.16)$$

For a detailed study of the (p, q) -binomial coefficients see [27]. This identity (4.16) leads to the following recurrence relation for $\psi_n(x; p, q)$:

$$\begin{aligned} \sqrt{[n+1]_{p,q}} \psi_{n+1}(x; p, q) &= \psi_n(px; p, q) + xp^n \psi_n(p^{-1}x; p, q) \\ &\quad - x(p - q) \sqrt{[n]_{p,q}} \psi_{n-1}(x; p, q). \end{aligned} \quad (4.17)$$

To obtain a realization of the (p, q) -oscillator algebra in the basis provided by $\{\psi_n(x; p, q) \mid n = 0, 1, 2, \dots\}$ let us proceed as follows. As before, define the number operator $\hat{\mathcal{N}}$ formally as

$$\hat{\mathcal{N}} \psi_n(x; p, q) = n \psi_n(x; p, q). \quad (4.18)$$

Note that

$$\hat{D}_{p,q}^{n+1} \psi_n(x; p, q) = 0, \quad (4.19)$$

and $\hat{D}_{p,q}^m \psi_n(x; p, q) \neq 0$ for any $m < n + 1$. Then, with the scaling operator defined by

$$\hat{\eta}_s f(x) = f(sx), \quad (4.20)$$

it readily follows from (4.15) and (4.17) that we can write

$$\hat{\mathcal{A}}_- \psi_n(x; p, q) = \hat{D}_{p,q} \psi_n(x; p, q) = \sqrt{[n]_{p,q}} \psi_{n-1}(x; p, q), \quad (4.21)$$

$$\begin{aligned} \hat{\mathcal{A}}_+ \psi_n(x; p, q) &= \left(\hat{\eta}_p + x \hat{\eta}_{p^{-1}} p^{\hat{\mathcal{N}}} - x(p - q) \hat{D}_{p,q} \right) \psi_n(x; p, q) \\ &= \sqrt{[n+1]_{p,q}} \psi_{n+1}(x; p, q), \end{aligned} \quad (4.22)$$

$$\hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- \psi_n(x; p, q) = [n]_{p,q} \psi_n(x; p, q) = [\hat{\mathcal{N}}]_{p,q} \psi_n(x; p, q), \quad (4.23)$$

$$\hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ \psi_n(x; p, q) = [n+1]_{p,q} \psi_n(x; p, q) = [\hat{\mathcal{N}} + 1]_{p,q} \psi_n(x; p, q), \quad (4.24)$$

which generalize the corresponding results (3.21)-(3.24) for the q -oscillator; when $p \rightarrow 1$, (4.21)-(4.24) reduce to (3.21)-(3.24). It is straightforward to verify that the realizations of $\{\hat{\mathcal{A}}_-, \hat{\mathcal{A}}_+, \hat{\mathcal{N}}\}$ in (4.18), (4.21) and (4.22) satisfy the required relations of the (p, q) -oscillator algebra (4.2). Using (4.21) and (4.22) in (4.23) we get the (p, q) -differential equation satisfied by $\psi_n(x; p, q)$ as

$$\left[(p - q)x \hat{D}_{p,q}^2 - \left(\hat{\eta}_p + p^{n-1} x \hat{\eta}_{p^{-1}} \right) \hat{D}_{p,q} + [n]_{p,q} \right] \psi_n(x; p, q) = 0, \quad (4.25)$$

which reduces to (3.25) in the limit $p \rightarrow 1$.

5 Conclusion

Let us conclude with a few remarks.

By choosing different values for p and q one can study different special cases of $\psi_n(x; p, q)$. This is particularly important since there exist many versions of q -oscillators which are special cases of the (p, q) -oscillator. For example, the q -oscillator originally used in connection with $U_q(su(2))$ [4]-[7], and more popular in physics literature, corresponds to the choice $p = q^{-1}$. From the above it is clear that this oscillator can be realized through the polynomials

$$H_n(x; q^{-1}, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{q^2} q^{-k(n-k)} x^k. \quad (5.1)$$

It may be emphasized again that though $H_n(x; q^{-1}, q)$ is a function with only a single q -parameter it cannot be rewritten as a regular Rogers-Szegö polynomial.

In [22] the raising and lowering operators for the Stieltjes-Wigert polynomial have been obtained using the fact that this polynomial is just the Rogers-Szegö polynomial with q replaced by q^{-1} and it has been shown that these raising and lowering operators of the Stieltjes-Wigert polynomial provide a realization of the single-parameter deformed oscillator with q replaced by q^{-1} . Now, it is clear that one can study the (p, q) -Stieltjes-Wigert polynomials similarly by replacing p and q , respectively, by p^{-1} and q^{-1} in the above formalism. Thus, the (p, q) -Stieltjes-Wigert polynomial is given by

$$G_n(x; p, q) = H_n(x; p^{-1}, q^{-1}) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pq)^{-k(n-k)} x^k, \quad (5.2)$$

which becomes the usual Stieltjes-Wigert polynomial in the limit $p \rightarrow 1$.

The continuous q -Hermite polynomial is defined as

$$H_n(\cos \theta | q) = e^{-in\theta} H_n(e^{2i\theta}; q). \quad (5.3)$$

It is clear that one can define the continuous (p, q) -Hermite polynomial in an analogous way as

$$H_n(\cos \theta | p, q) = e^{-in\theta} H_n(e^{2i\theta}; p, q). \quad (5.4)$$

This has already been suggested in [13] without any further study. It should be worthwhile to study the (p, q) -Rogers-Szegö polynomial, the (p, q) -Stieltjes-Wigert polynomial and the continuous (p, q) -Hermite polynomial in detail.

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